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ON CROSSINGS OF GAUSSIAN FIELDS

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A formula is proved for the expectation of the $(d-1)$-dimensional measure of the intersection of a Gaussian stationary random field with a fixed level u.

Crossings, Gaussian, stationary random field, perimeter

Let O be open and A a Borel set in R^d. We shall denote the *perimeter of A relative to O* [6] by

$$\Pi_O(A) = \sup\left\{ \int_A \operatorname{div}(u)\, dt : u \in (C_k^\infty(O))^d, \|u(t)\| \leq 1 \quad \forall t \in R^d \right\}. \tag{1}$$

Here $(C_k(O))^d$ denotes the C^∞-functions with compact support from O to R^d, $\|\cdot\|$ the Euclidean norm and dt the Lebesgue measure in R^d.

We shall say that the perimeter of A relative to O is finite (resp. infinite) if $\Pi_O(A)$ is finite (resp. infinite).

If the boundary ∂A of A is regular enough as to apply Green's formula, we have

$$\int_A \operatorname{div}(u)\, dt = \int_{\partial A} (u, n)\, d\sigma_{d-1} \tag{2}$$

where $(\cdot,\cdot)$ denotes the usual scalar product in R^d, n the outer normal to ∂A and σ_{d-1} the $(d-1)$-dimensional area of the surface ∂A.

It is clear from (1) and (2) that for regular ∂A one has that

$$\Pi_O(A) = \sigma_{d-1}(\partial A \cap O).$$

The following lemma is not difficult to prove (see [6], for example) and is useful to compute perimeters.

Lemma 1. (i) *Let $\{g_n\}$ be a uniformly bounded sequence of real measurable functions that converges almost everywhere (with respect to Lebesgue measure) to the indicator*

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function χ_A of the set A. Then

$$\Pi_O(A) \leq \liminf_n \int_O \|\operatorname{grad} g_n(t)\| \, dt. \tag{3}$$

(ii) *For every $\varepsilon > 0$, let $\Psi_\varepsilon : R^d \to R$ be a C^∞ even convolution kernel with support contained in $\{t : \|t\| < \varepsilon\}$ (that is $\Psi_\varepsilon(t) \geq 0$, $\int_{R^d} \Psi_\varepsilon(t)\, dt = 1$), and for every function $f \in L^1_{\text{loc}}(R^d)$, put*

$$f_\varepsilon(t) = (\Psi_\varepsilon * f)(t) = \int_{R^d} \Psi_\varepsilon(t-s) f(s)\, ds. \tag{4}$$

Then, if $0 < \varepsilon < \delta$, one has

$$\int_{O_{-\delta}} \|\operatorname{grad}(\chi_A)_\varepsilon(t)\| \, dt \leq \Pi_O(A) \tag{5}$$

where $O_{-\delta} = \{t : \operatorname{dist}(t, \bar{O}^c) > \delta\}$ ($\operatorname{dist}(t, \bar{O}^c)$ is the Euclidean distance from t to the complement of the closure of the set O).

Consider now a d-dimensional parameter Gaussian, separable and centered process $\{X(t) : t \in R^d\}$ with covariance function

$$\Gamma(t) = \mathbf{E}(X(0)X(t)),$$

normalized by $\Gamma(0) = 1$. We shall suppose that the process has continuous realizations.

We are interested in the random crossing set

$$C_u = \{t : t \in R^d, X(t) = u\}$$

of the process with the level u, and, more precisely, in the set $C_u \cap T$, where

$$T = (0, 1)^d.$$

(The choice of this subset of the parameter set to look at the process is unimportant for what follows.)

The continuity of the realizations plus the fact that, with probability one, the process does not have local extrema on the level u (this has been proved for $d = 1$ by Ylvisaker [8]; his proof can be extended to the case $d \geq 2$ with minor changes), imply that, with probability one,

$$C_u = \partial A_u = \partial B_u \tag{6}$$

where A_u and B_u are the random open (with probability one) sets,

$$A_u = \{t : t \in R^d, X(t) < u\}, \qquad B_u = \{t : t \in R^d, X(t) > u\}.$$

For regular processes the expectation of $\sigma_{d-1}(C_u \cap T)$ has been computed by Benzaquen and Cabaña [4] for processes with Γ six times differentiable and by the present author when the realizations are two times continuously differentiable [7]. The proofs depend heavily on the local properties of the realizations of regular

Gaussian fields (Belyaiev [3] and Adler and Hasofer [1], [2] for the first proof, and also on the Landau–Shepp inequality for the second).

The theorem below extends the formula to the general case, looking at C_u as the boundary of A_u (see (5)), and computing the expectation of $\Pi_T(A_u)$, the perimeter of A_u relative to T. Additionally the proof becomes simpler than the former ones in the cases mentioned above.

For $d=1$ the formula in Theorem 1 has been well known in its full generality since some time ago (see [5]); in this case it is easy to prove that $\Pi_T(A_u) = \#(C_u \cap T)$ whether it is finite or infinite.

Let us still introduce some standard notation before the statement of the theorem. Let μ_Γ be the spectral (probability) measure of the process $\{X(t): t \in \boldsymbol{R}^d\}$, that is,

$$\Gamma(t) = \int_{\boldsymbol{R}^d} \exp\{\mathrm{i}(t, s)\}\, \mathrm{d}\mu_\Gamma(s), \quad t \in \boldsymbol{R}^d,$$

and put

$$v = \int_{\boldsymbol{R}^d} \|s\|^2\, \mathrm{d}\mu_\Gamma(s)$$

(v may be finite or infinite).

Γ is twice differentiable at the origin if and only if v is finite and in this case the second partial derivatives of Γ are continuous functions and we have that

$$\left(\left(\frac{\partial^2 \Gamma}{\partial t_i\, \partial t_j}\right)\right) = -\int_{\boldsymbol{R}^d} ss^* \exp(\mathrm{i}(t, s))\, \mathrm{d}\mu_\Gamma(s) \tag{7}$$

(s^* denotes the transposed of the column vector s and t_i the ith coordinate of t).

Put also

$$\boldsymbol{R} = -\left(\left(\frac{\partial^2 \Gamma}{\partial t_i \partial t_j}\bigg|_{t=0}\right)\right) = \int_{\boldsymbol{R}^d} ss^*\, \mathrm{d}\mu_\Gamma(s).$$

Theorem 1. (i) *If $v < \infty$ and $\boldsymbol{R}$ is non-singular, then*

$$\mathbf{E}(\Pi_T(A_u)) = \phi(u)\mathbf{E}(\|\xi\|) \tag{8}$$

where $\phi(u) = (2\pi)^{-1/2} \exp\{-\frac{1}{2}u^2\}$ is the standard (0, 1) *normal density function and ξ is a centered d-dimensional Gaussian vector,* $\mathrm{var}(\xi) = \boldsymbol{R}$.

(ii) *If $v = \infty$, we have that*

$$\mathbf{E}(\Pi_T(A_u)) = +\infty.$$

(*Note*: The additivity of the left-hand side of (8) as a function of T implies that if we substitute $T = (0, 1)^d$ by another open set, we must simply multiply on the right-hand side by its Lebesgue measure to obtain the expectation of $\Pi_T(A_u)$.)

Proof. We first prove (i).

Denote by h_u the indicator function of the open half line $(-\infty, u)$ of the real numbers. Clearly

$$\chi_{A_u}(t) = h_u(X(t)).$$

Let $\{f_m\}$ be an increasing sequence of nonincreasing C^∞-functions defined on the real line, such that

$$f_m(x) = \begin{cases} 1 & \text{for } x < u - 1/m, \\ 0 & \text{for } x \geq u. \end{cases}$$

It is obvious that $f_m \to h_u$ when $m \to \infty$, and that with the notation of (4), with probability one,

$$f_m(X_\rho(t)) \to h_u(X(t))$$

for almost every t when $m \to \infty$ and $\rho \to 0$, since C_u has zero Lebesgue-measure with probability one. Hence, according to Lemma 1(i), Fatou's lemma and the stationarity of the process we have that

$$\begin{aligned} \mathbf{E}(II_T(A_u)) &\leq \liminf \mathbf{E}\left\{ \int_T \|\mathrm{grad}\, f_m(X_\rho(t))\| \, \mathrm{d}t \right\} \\ &= \liminf \mathbf{E}\{\|\mathrm{grad}\, f_m(X_\rho(0))\|\}. \end{aligned} \tag{9}$$

But

$$\mathbf{E}\{\|\mathrm{grad}\, f_m(X_\rho(0))\|\} = \mathbf{E}\{|f'_m(X_\rho(0))| \, \|\mathrm{grad}\, X_\rho(0)\|\}. \tag{10}$$

Since $X_\rho(0)$ and grad $X_\rho(0)$ are independent—due to the fact that the first differential of the covariance function, if it exists, must be zero at the origin—the right-hand member of (10) equals

$$\mathbf{E}\{|f'_m(X_\rho(0))|\}\mathbf{E}\{\|\mathrm{grad}\, X_\rho(0)\|\} \to \int_R |f'_m(x)|\phi(x)\, \mathrm{d}x E(\|\xi\|) \quad \text{for } \rho \to 0 \tag{11}$$

where ξ is as in the statement of the theorem.

On the other hand, $|f'_m(x)|\mathrm{d}x$ converges weakly to the unit atom at $x = u$ as $m \to \infty$, so that (9) together with (11) implies that

$$\mathbf{E}(II_T(A_u)) \leq \phi(u)\mathbf{E}(\|\xi\|). \tag{12}$$

For the converse inequality, fix $\delta > 0$ and choose $0 < \varepsilon < \delta$. Using Lemma 1(ii) and the stationarity of the process, we get

$$\begin{aligned} \mathbf{E}(II_T(A_u)) &\geq \mathbf{E}\left\{ \int_{T_\delta} \|\mathrm{grad}(\chi_{A_u})_\varepsilon(t)\| \, \mathrm{d}t \right\} \\ &= (1 - 2\delta)^d \mathbf{E}\{\|\mathrm{grad}(\chi_{A_u})_\varepsilon(0)\|\}. \end{aligned} \tag{13}$$

Take again $\{f_m\}$ as before. Then, applying dominated convergence, we have that

$$\operatorname{grad}(\chi_{A_u})_\varepsilon(0) = -\int_{R^d} \operatorname{grad} \Psi_\varepsilon(s) h_u(X(s))\,\mathrm{d}s$$

$$= -\lim_{\substack{m\to\infty\\ \rho\to 0}} \int_{R^d} \operatorname{grad} \Psi_\varepsilon(s) f_m(X_\rho(s))\,\mathrm{d}s$$

and

$$\mathbf{E}\{\|\operatorname{grad}(\chi_{A_u})_\varepsilon(0)\|\} = \lim_{\substack{m\to\infty\\ \rho\to 0}} \mathbf{E}\left\{\left\|\int_{R^d} \operatorname{grad} \Psi_\varepsilon(s) f_m(X_\rho(s))\,\mathrm{d}s\right\|\right\}$$

$$= \lim_{\substack{m\to\infty\\ \rho\to 0}} \mathbf{E}\left\{\left\|\int_{R^d} \Psi_\varepsilon(s) \operatorname{grad} f_m(X_\rho(s))\,\mathrm{d}s\right\|\right\}$$

$$= \lim_{\substack{m\to\infty\\ \rho\to 0}} \mathbf{E}\left\{\left\|\int_{R^d} \Psi_\varepsilon(s) f'_m(X_\rho(s)) \operatorname{grad} X_\rho(s)\,\mathrm{d}s\right\|\right\}. \tag{14}$$

On account of the fact that f'_m has constant sign, the expectation in the right-hand side of (14) can be bounded from below by

$$\mathbf{E}\left\{\int_{R^d} \Psi_\varepsilon(s)|f'_m(X_\rho(s))|\|\operatorname{grad} X_\rho(0)\|\,\mathrm{d}s\right\} +$$

$$-\mathbf{E}\left\{\int_{R^d} \Psi_\varepsilon(s)|f'_m(X_\rho(s))|\|\operatorname{grad} X_\rho(s) - \operatorname{grad} X_\rho(0)\|\,\mathrm{d}s\right\}. \tag{15}$$

The first term in (15) is

$$\int_{R^d} \Psi_\varepsilon \mathbf{E}\{|f'_m(X_\rho(s))|\|\operatorname{grad} X_\rho(0)\|\}\,\mathrm{d}s,$$

which tends to

$$\int_{R^d} \Psi_\varepsilon(s)\,\mathrm{d}s \int_{R^{d+1}} |f'_m(x)|\,\|y\| p_s(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{16}$$

as $\rho \to 0$, where $p_s(x, y)$ is the (normal) density function of the pair $(X(s), \operatorname{grad} X(0))$.

Letting first $m \to \infty$ in (16) and $\varepsilon \to 0$ afterwards, repeated application of Fatou's lemma implies that the lim inf of the first term in (15) is greater than or equal to

$$\int_{R^d} \|y\|\, p_0(u, y)\,\mathrm{d}y = \phi(u)\mathbf{E}(\|\xi\|), \tag{17}$$

using again the fact that $X(0)$ and $\operatorname{grad} X(0)$ are independent.

Let us now look at the second term in (15). It is obviously bounded by

$$\sum_{j=1}^{d} \mathbf{E}\left\{\int_{R^d} \Psi_\varepsilon(s)|f'_m(X_\rho(s))| \left|\frac{\partial X_\rho}{\partial t_j}(s) - \frac{\partial X_\rho}{\partial t_j}(0)\right|\,\mathrm{d}s\right\}. \tag{18}$$

The jth term in this sum is equal to

$$\int_{R^d} \Psi_\varepsilon(s)\,\mathrm{d}s \int_{R^2} |f'_m(x)|\,|z|\,p_{j,s,\rho}(x,z)\,\mathrm{d}x\,\mathrm{d}z \tag{19}$$

where $p_{j,s,\rho}$ is the normal, centered density function with covariance matrix

$$\begin{pmatrix} \Gamma_\rho(0) & \dfrac{\partial\Gamma_\rho}{\partial t_j}(s) \\ \dfrac{\partial\Gamma_\rho}{\partial t_j}(s) & 2\left\{\dfrac{\partial^2\Gamma_\rho}{\partial t_j^2}(s) - \dfrac{\partial^2\Gamma}{\partial t_j^2}(0)\right\} \end{pmatrix}.$$

Here Γ_ρ is the covariance function of the regularized process $\{X_\rho(t): t \in R^d\}$. One can easily verify that it has spectral measure

$$\left|\int_{R^d} \Psi_\rho(t)\exp\{\mathrm{i}\langle s,t\rangle\}\,\mathrm{d}t\right|^2 \mathrm{d}\mu_\Gamma(s).$$

The following lemma, which can be proved by direct computation is useful to bound the expression in (19).

Lemma 2. *Let $p(x,y)$ be a centered two-dimensional Gaussian density, corresponding to the covariance matrix*

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

and assume that $\frac{1}{2} \leq \sigma_{11} \leq 1$. Then we have that

$$\int_{-\infty}^{\infty} |y|\,p(x,y)\,\mathrm{d}y \leq k\sigma_{22}^{1/2}|x|\exp\{-Lx^2\}$$

where k and L are positive constants, independent of the σ_{ij}'s.

Proof of Theorem 1 (continued). Since $\Gamma_\rho(0)$ is near 1 when ρ is small enough, the lemma can be applied to obtain the following bound of (19):

$$k\int_{R^d} \Psi_\varepsilon(s)\,\mathrm{d}s\left\{\int_{R^1} |f'_m(x)|\left\{2\left\{\frac{\partial^2\Gamma_\rho}{\partial t_j^2}(s) - \frac{\partial^2\Gamma_\rho}{\partial t_j^2}(0)\right\}\right\}^{1/2} |x|\exp\{-Lx^2\}\,\mathrm{d}x\right\}$$

which tends to zero if $\rho \to 0$, $m \to \infty$ and $\varepsilon \to 0$ in that order, since $\partial^2\Gamma/\partial t_j^2$ is a continuous function of s.

Summing up, this implies that $\mathbf{E}(\Pi_T(A_u)) \geq \phi(u)\mathbf{E}(\|\xi\|)$ and hence, finishes part (i) of the proof of the theorem.

Let us now turn to part (ii). We may suppose with no loss of generality that

$$\int_{R^d} t_1^2\,\mathrm{d}\mu_\Gamma(t) = \infty. \tag{20}$$

We use again Lemma 1(ii), so that

$$\Pi_T(A_u) \geq \int_{T_{-\delta}} \|\mathrm{grad}(\chi_{A_u})_\varepsilon(s)\|\,\mathrm{d}s \tag{21}$$

for $0<\varepsilon<\delta<\frac{1}{2}$, but now we shall choose the kernel Ψ_ε of the form,

$$\Psi_\varepsilon(t) = g_1(t_1)g_2(t_2, \ldots, t_d)$$

where g_1 and g_2 are nonnegative C^∞-functions defined on R^1 and R^{d-1}, with support contained in $\{|t_1|<\varepsilon_1\}$ and $\{(t_2^2+\cdots+t_d^2)^{1/2}<\varepsilon_2\}$, $\varepsilon_1^2+\varepsilon_2^2<\varepsilon^2$, respectively, and such that

$$\int_{R^1} g_1\,\mathrm{d}t_1 = \int_{R^d} g_2\,\mathrm{d}t_2\cdots\mathrm{d}t_d = 1.$$

Then,

$$\begin{aligned}
&\frac{\partial}{\partial t_1}(\chi_{A_u})_\varepsilon(s) = \\
&= \int_{R^d} \frac{\partial}{\partial t_1}\Psi_\varepsilon(s-t)h_u(X(t))\,\mathrm{d}t \\
&= \int_{R^1} g_1'(s_1-t_1)\,\mathrm{d}t_1 \int_{R^{d-1}} g_2(s_2-t_2, \ldots, s_d-t_d)h_u(X(t))\,\mathrm{d}t_2\cdots\mathrm{d}t_d.
\end{aligned} \tag{22}$$

As a consequence of the fact that C_u has zero Lebesgue measure with probability one, the set

$$M = \{(s_2, \ldots, s_d)\colon \mu_1(\{t_1\colon (t_1, s_2, \ldots, s_d)\in C_u\})>0\}$$

has zero μ_{d-1}-measure.

Take $(s_2, \ldots, s_d)$ *not* belonging to M. Excepting the null measure set of t_1's such that $(t_1, s_2, \ldots, s_d)\in C_u$, one has that

$$\begin{aligned}
&\int_{R^{d-1}} g_2(s_2-t_2, \ldots, s_d-t_d)h_u(X(t_1, t_2, \ldots, t_d))\,\mathrm{d}t_2\cdots\mathrm{d}t_d \to \\
&\qquad \to h_u(X(t_1, s_2, \ldots, s_d)) \quad \text{for } \varepsilon_2\to 0,
\end{aligned}$$

since $h_u(X(t_1, s_2, \ldots, s_d))$ is a continuous function of $s_2, \ldots, s_d$ when $(t_1, s_2, \ldots, s_d)\notin C_u$. As the inner integral in (22) is obviously uniformly bounded by 1, one may apply dominated convergence when $\varepsilon_2\to 0$, thus obtaining

$$\lim_{\varepsilon_2\to 0}\frac{\partial}{\partial t_1}(\chi_{A_u})_\varepsilon(s) = \int_{R^1} g_1'(s_1-t_1)h_u(X(t_1, s_2, \ldots, s_d))\,\mathrm{d}t_1 \tag{23}$$

for $(s_2, \ldots, s_d) \notin M$ and any s_1. So (23) holds almost everywhere in R^d. From (21) we then get that

$$\Pi_T(A_u) \geq \int_{T_\delta} \left| \frac{\partial}{\partial t_1} (\chi_{A_u})_\varepsilon(s) \right| \mathrm{d}s \to$$

$$\to \int_{T_\delta} \left| \int_{R^1} g_1'(s_1 - t_1) h_u(X(t_1, s_2, \ldots, s_d))\, \mathrm{d}t_1 \right| \mathrm{d}s \quad \text{for } \varepsilon_2 \to 0 \tag{24}$$

on account of (23) and the bound

$$\left| \frac{\partial}{\partial t_1} (\chi_{A_u})_\varepsilon(s) \right| \leq \int_{R^1} |g_1'(s_1)|\, \mathrm{d}s_1$$

which is finite for fixed $\varepsilon_1 > 0$.

The limit in (24) can be written as

$$\int_\delta^{1-\delta} \cdots \int_\delta^{1-\delta} \mathrm{d}s_2 \cdots \mathrm{d}s_d \int_\delta^{1-\delta} \left| \int_{R^1} g_1'(s_1 - t_1) h_u(X(t_1, s_2, \ldots, s_d))\, \mathrm{d}t_1 \right| \mathrm{d}s_1. \tag{25}$$

For fixed $(s_2, \ldots, s_d) \notin M$ introduce now the (one-dimensional) process

$$Y(t_1) = X(t_1, s_2, \ldots, s_d)$$

and put

$$A_u^{s_2, \ldots, s_d} = \{t_1 : Y(t_1) < u\}.$$

It is clear that Y is a stationary Gaussian process with covariance function

$$\Gamma_Y(t_1) = \Gamma(t_1, 0, \ldots, 0),$$

so that, for any open non-empty interval J of the real numbers

$$\mathbf{E}(\Pi_J(A_u^{s_2, \ldots, s_d})) = +\infty \tag{26}$$

because of (20) and the validity of the theorem for $d = 1$.

On the other hand,

$$\int_{R^1} g_1(s_1 - t_1) h_u(Y(t_1))\, \mathrm{d}t_1 \to h_u(Y(s_1)) \quad \text{for } \varepsilon_1 \to 0$$

for almost every $s_1 \in R^1$.

Applying now Lemma 1(i) it turns out that the lim inf when $\varepsilon_1 \to 0$ of the inner integral in (25) is not smaller than

$$\Pi_{(\delta, 1-\delta)}(A_u^{s_2, \ldots, s_d}).$$

Taking expectations one gets

$$\mathbf{E}(\Pi_T(A_u)) \geq \int_\delta^{1-\delta} \cdots \int_\delta^{1-\delta} \mathbf{E}(\Pi_{(\delta, 1-\delta)}(A_u^{s_2, \ldots, s_d}))\, \mathrm{d}s_2 \cdots \mathrm{d}s_d = +\infty$$

according to (26).

References

[1] R.J. Adler, Excursions above a fixed level by n-dimensional random fields, J. Appl. Probab. 13 (1976) 276–289.

[2] R.J. Adler and A.M. Hasofer, Level crossings for random fields, Ann. Probab. 4 (1976) 1–12.

[3] Y. Belyayev, Point processes and first passage problems, Proc. 6th Berkeley Symp. Math. Stat. Prob. 3 (1972) 1–17.

[4] S. Benzaquen and E [illegible]ña, The expected measure of the level set of a regular stationary Gaussian process, Pac[illegible] Math., to appear.

[5] H. Cramèr and M.R. Leadbetter, Stationary and Related Stochastic Processes (Wiley, New York 1967).

[6] M. Miranda, Frontière minime, Mon. Matemat. No. 27 (IMPA, Rio de Janeiro, 1976).

[7] M. Wschebor, Sobre las superficies de nivel de un proceso Gaussiano estacionario de parámetro d-dimensional, Actas de 4° SINAPE, Rio de Janeiro (1980).

[8] D. Ylvisaker, The expected number of zeros of a stationary Gaussian process, Ann. Math. Statist. (1965) 1043–1046.